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Numerical solution to the hermitian Yang-Mills equation on the Fermat quintic

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Abstract: We develop an iterative method for finding solutions to the hermitian Yang-Mills equation on stable holomorphic vector bundles, following ideas recently developed by Donaldson. As illustrations, we construct numerically the hermitian Einstein metrics on the tangent bundle and a rank three vector bundle on $\mathbb{P}^2$. In addition, we find a hermitian Yang-Mills connection on a stable rank three vector bundle on the Fermat quintic.

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1. Introduction

The modern study of compactification of higher dimensional theories can be divided into two general branches. The first makes use of compactification manifolds with a good deal of symmetry, such as the torus, sphere, squashed spheres and so on. Such spaces have explicitly known metrics, allowing explicit solutions of the equations of motion, and explicit Kaluza-Klein reduction. These solutions have many applications, such as supergravity duals of large N gauge theories; however their high degree of symmetry tends to be a problem in trying to obtain models with the level of complexity of the Standard Model or its often-postulated extensions.

The second branch makes use of manifolds for which the relevant metrics are known to exist by general theorems, but for which explicit expressions are not known. The most famous examples are the Ricci-flat Kahler metrics conjectured to exist by Calabi and proven to exist by Yau [1]. In 1985, it was proposed by Candelas et al [2] that compactification of the heterotic string on a Calabi-Yau manifold could lead to quasi-realistic theories of particle physics, containing grand unified extensions of the Standard Model and low energy

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supersymmetry. Since then, other metrics of this type, such as G_2 holonomy metrics, have been used in quasi-realistic compactifications; see for example [3].

Over the subsequent years, many tricks were developed to bypass the difficulties posed by not knowing the compactification metric. These tricks began with the algebraic geometry behind the theorems of Yau and Donaldson-Uhlenbeck-Yau, and gradually evolved into entire branches of mathematical physics, such as topological string theory and special geometry. To drastically oversimplify, the general picture is that certain "protected" quantities in the four dimensional Effective Field Theory (EFT), such as the superpotential in theories with four supercharges, and the prepotential in theories with eight supercharges, can be computed using techniques combining algebraic geometry with physical ideas. Other quantities, such as the Kahler potential in theories with four supercharges, cannot be computed directly. Since a good deal of important physics depends on the Kahler potential — precise values of particle masses, and the existence and stability of supersymmetry breaking vacua, this situation is not very satisfactory.

Almost all present knowledge about the Kahler potential in the EFT comes from studying expansions around more computable limits. The best known example is the case of $N = 1$ compactifications which contain $N = 2$ subsectors, such as heterotic $(2, 2)$ models, or type II on Calabi-Yau orientifolds. In these cases, there is a limit in which part of the $N = 1$ Kahler potential becomes equal to that of the related $N = 2$ theory, which is computable using special geometry. Other examples include the solvable orbifold or Gepner model limits, at which the entire Kahler potential is computable in principle using CFT techniques. However, it is not clear at present how representative such results are of the general case. Even a limited ability to compute in the general case would allow studying this question.

One completely general technique for addressing such problems is to compute the Ricci-flat metrics and related quantities numerically. Numerical methods are unavoidable in other areas of physics, beginning with such seemingly elementary problems as computing the spectrum of the helium atom or integrating Newton's equations for the three body problem in celestial mechanics; it would be surprising if string theory could avoid this. To bring string theory closer to a possible confrontation with real data, for example from collider physics, it may be valuable to develop these missing parts of the theory of compactification.

In this work, we make a start in this direction by showing two things. First, we review how to use existing mathematical techniques to numerically approximate metrics on Kahler manifolds, along lines recently developed by Donaldson [4]. Second, we extend these mathematical techniques to hermitian Yang-Mills connections. It will be clear that these techniques could be pushed to compute higher order terms, metrics on moduli spaces, and the like. A subsequent paper will explain the numerical methods in more detail and do some simple computations of terms in the EFT for compactification on a quintic Calabi-Yau 3-manifold.

Our direct inspirations are Donaldson's work [4] on numerical approximation of metrics, and of Wang [5] developing the corresponding mathematics for vector bundles. We can also cite Headrick and Wiseman [6], who made a pioneering numerical study of the K3 metric using position-space methods. Finally, the first author is particularly indebted to

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Bernie Shiffman and Steve Zelditch for teaching him the basics of asymptotic analysis on holomorphic line bundles, and for advice in the early stages of this project, in particular for pointing out Wang's work.

Let us briefly explain the problem and survey some of the approaches one might take towards it, before beginning the detailed development in section 2. Following [2], the derivation of the matter Lagrangian in a heterotic compactification on a Calabi-Yau X carrying a bundle V involves the following steps:

(1) Find the Ricci-flat metric g_{ij} (with specified moduli) on X.

(2) Find the hermitian Yang-Mills connection A_i on V.

(3) Find the zero modes ψ^α of the Dirac operator. As is standard, on a Kahler manifold this amounts to finding harmonic differential forms ψ valued in V, i.e. solutions of $0 = (\bar\partial + \bar A)\psi = (\bar\partial + \bar A)^*\psi$, where $*$ denotes the adjoint operator.

(4) Find an orthonormal basis of forms ψ.

(5) Compute the integrals over X of wedge products of these forms to get the superpotential.

The key step for us is (4). Existing methods for computing the superpotential, such as [7–9], accomplish step (5) without needing the results of (1) and (2), by using unnormalized zero modes. This leads to a superpotential defined in terms of fields whose kinetic term is obtained from "some" unknown Kahler potential. To do better, we must either derive normalized zero modes in (4) for use in (5), or else take the zero modes used in (5) and compute their normalizations using the explicit metric from (1).

There seems to be no way of doing this without some knowledge of the Ricci-flat metric and thus the first step is to choose some approximation scheme for this metric. One's first thought might be to follow standard practice in numerical relativity, as done in [6], and introduce a six dimensional lattice which is a discrete approximation to the manifold X; in other words a position space approach. Taking the Kahler potential K as the basic dynamical variable, Einstein's equations reduce to the complex Monge-Ampere equation

$$\det(\partial\bar\partial K) = \Omega \wedge \bar\Omega \tag{1.1}$$

which can be solved by relaxation methods. One would then need to find similar lattice approximations for the connection on V and the zero modes.

An alternative approach, introduced by Donaldson [4], is to use the natural embedding of X into $\mathbb{P}^{N-1}$ provided by the N sections of an ample line bundle L^k (we will explain this in detail below). We then take as a candidate approximating metric on X the pull-back of a Fubini-Study metric on $\mathbb{P}^{N-1}$. Such a metric is defined by an $N \times N$ hermitian matrix. By suitably choosing this matrix we can try to make the associated Fubini-Study metric restrict to X in such a way that it gives a good approximation to the Ricci-flat metric on X.

A major advantage of this approach is that it avoids the complications and arbitrariness involved in choosing an explicit discretization of X; rather the entire approximation scheme

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follows from a single parameter k, the scale of the first Chern class of L. Subsequent mathematical development reveals more structure which can be used to our advantage. For example, a very natural approximation to the Ricci-flat metric, which becomes exact as $k \to \infty$, is the so-called "balanced" metric. In a sense, to be described below, this is the metric for which the embedding of X into $\mathbb{P}^{N-1}$ has its center of mass at the "origin". It also satisfies a simple fixed point condition which can be used for relaxation, solving step (1).

Another advantage, which is key for the present application, is that Donaldson's method can be naturally extended to study holomorphic vector bundles on X. There is a standard relation between holomorphic connections and hermitian metrics, which we review in section 2, in which step (2) of the above prescription is turned into the problem of finding a hermitian-Einstein metric on a vector bundle. For illustrative purposes we will explicitly study hermitian-Einstein metrics on two spaces: complex projective space $\mathbb{P}^n$ and the Fermat quintic threefold.

The organization of the paper is as follows. In section 2 we provide an overview of the geometric background needed for our construction, in particular we will describe Donaldson's approach for getting metrics of constant scalar curvature. In section 3 we explain a numerical approximation to the hermitian Einstein metric on a holomorphic vector bundle by a simple adaptation of Donaldson's scheme, building on mathematical work of Wang. In section 4 we focus on several explicit examples. Here we describe some of our numerical methods and results in detail. By design we are also able to test our approximation scheme for $T\mathbb{P}^2(k)$, where $T\mathbb{P}^2$ is the holomorphic tangent bundle of $\mathbb{P}^2$, since in this case one has an analytic solution.

2. Metrics of constant scalar curvature

We follow the plan outlined in the introduction, beginning with step (1). Let X be an n-dimensional complex Kahler "compactification manifold." Since we are not assuming it is a valid string theory background, we can generalize the discussion to arbitrary n and first Chern class $c_1(X)$.

The basic example we have in mind is the complex projective space $\mathbb{P}^n$, parameterized by the standard homogenous coordinates $\{Z_i\}_{i=0}^n$, up to the identification $\{Z_i\} \cong \{\lambda Z_i\}$ for $\lambda \in \mathbb{C}^*$. The Kahler potential

$$K_{FS} = k \ \log \left(\sum_{i=0}^{n} |Z_i|^2 \right) \tag{2.1}$$

defines the Fubini-Study metric on $\mathbb{P}^n$, with $\mathrm{SU}(n+1)$ symmetry $Z_i \to g_i^j Z_j$, and $g \in \mathrm{U}(n+1)$. The parameter k controls the Kahler class $\omega = \partial\bar{\partial} K_{FS}$.

Our other general example is the hypersurface defined by the vanishing of a degree d polynomial in $\mathbb{P}^n$:

$$f(Z) = \sum_{i_1 \cdots i_d} c^{i_1 \cdots i_d} Z_{i_1} \ldots Z_{i_d}. \tag{2.2}$$

For $n = 4$ and $d = 5$ we get a quintic threefold Q. Its complex structure is determined by the 126 parameters $c^{i_1 \cdots i_5}$, modulo the action of $GL(5, \mathbb{C})$ on the Z_i's, which leaves 101

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parameters. The generic member of this family is smooth, and has $b^{1,1} = 1$. Therefore the Kahler class is determined by a single real number. A simple one parameter family of Kahler metrics on Q is obtained by pulling back the Fubini-Study metric on $\mathbb{P}^4$, or equivalently interpreting eq. (2.1) as a Kahler potential on Q. Of course this will not be Ricci-flat.

2.1 Approximating Ricci-flat metrics by projective embedding

We now want to find a larger space of Kahler metrics in which to find a better approximation to the Ricci flat metric. One simple generalization of eq. (2.1) can be obtained by choosing an $(n+1) \times (n+1)$ hermitian matrix $h^{i\bar{j}}$, and writing

$$K_h = k \log \left(\sum_{i,\bar{j}=0}^{n} h^{i\bar{j}} Z_i \bar{Z}_j \right). \tag{2.3}$$

Of course, by making a linear redefinition of coordinates, we could turn this back into eq. (2.1), but doing so would modify eq. (2.2). Rather, by fixing eq. (2.2), this way we get an $(n+1)^2$-parameter family of Kahler potentials.

Another way to think about this definition is to make the linear redefinition taking h to the identity. In this case, the parameters we are varying to control the metric are the extra 25 parameters in eq. (2.2) determining a specific embedding of Q into $\mathbb{P}^4$. While all of the embeddings are equivalent under a $GL(5,\mathbb{C})$ action, once we use the metric, we break this to $\mathrm{U}(5)$; thus the set of metrics we can obtain this way is parameterized by a $GL(5,\mathbb{C})/\mathrm{U}(5)$ homogeneous space.

A simple generalization to get more parameters could be motivated by noticing that eq. (2.1) is also equal to

$$K_{FS,k} = \log \left(\sum_{i=0}^{n} |Z_i|^2 \right)^k \tag{2.4}$$

$$= \log \left(\sum_{i_1,\cdots,i_k=0}^{n} Z_{i_1} \cdots Z_{i_k} \bar{Z}_{i_1} \cdots \bar{Z}_{i_k} \right) \tag{2.5}$$

and generalizing this to

$$K_{h,k} = \log \left(\sum_{i_1,\cdots,i_k,\bar{j}_1,\cdots,\bar{j}_k=0}^{n} h^{i_1\cdots i_k \bar{j}_1 \cdots \bar{j}_k} Z_{i_1} \cdots Z_{i_k} \bar{Z}_{\bar{j}_1} \cdots \bar{Z}_{\bar{j}_k} \right), \tag{2.6}$$

which can again be interpreted as a Kahler potential on Q. In simple terms, we are using higher degree polynomials as basis functions. Now we have an $(n+1)^{2k}$-parameter family of metrics, and by taking k large we can imagine finding an arbitrarily good approximating metric within this class.

One way to find the best approximation to the Ricci-flat metric on Q would be to write eq. (1.1) directly in these variables. Note that the holomorphic $(n,0)$-form Ω is

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known explicitly. For example, in the coordinate patch where $Z_0 \neq 0$ we can choose the local coordinates $w_i = Z_i/Z_0$, in terms of which

$$\Omega = \frac{dw_1\, dw_2\, dw_3}{\partial f/\partial w_4},$$

and thus one can write the volume form for the Ricci flat metric explicitly,

$$d\mathrm{vol}_X = \Omega \wedge \bar{\Omega} \tag{2.7}$$

without solving any equations. One might then substitute eq. (2.6) into eq. (1.1), evaluate this at a set of points p_i, and solve the resulting system of nonlinear equations. These are rather complicated, however, and furthermore we have introduced arbitrariness in the choice of the p_i. Now this arbitrariness can have its uses, for example we might use it to place more points in regions of large curvature. On the other hand, it means that the results will not have simple mathematical or physical interpretations, except in the limit in which the number of points is so large that we can ignore the discretization.[1] Before investing a lot of effort into their study, we should try to improve on this point.

2.2 Balanced metrics

There is a pretty construction that goes back to [10] which provides a more natural approximating metric, and a numerical scheme which is guaranteed to converge to it.

First, we can systematize the construction which led to eq. (2.6), by noting that the basis functions are products of degree k holomorphic times degree k antiholomorphic monomials. Let the number of independent holomorphic degree k monomials be $N+1$; this is the binomial coefficient $\binom{n+k}{k}$ for $\mathbb{P}^n$, and we will give it for Q later.

Let us phrase this construction in a way which can be used for an arbitrary manifold X. We choose a holomorphic line bundle $\mathcal{L}$ over X, with N global sections. Denote a complete basis of these as s_α, where $1 \leq \alpha \leq N$, and consider the map

$$i_k \colon X \longrightarrow \mathbb{P}^{N-1} \qquad i_k(Z_0, \ldots, Z_n) = (s_1(Z), s_2(Z), \ldots, s_N(Z)).$$

The geometric picture is that each point in our original manifold X (parameterized by the Z_i) corresponds to a point in $\mathbb{C}^N$ parameterized by the sections s_α. Since choosing a different frame for $\mathcal{L}$ would produce an overall rescaling $s_\alpha \to \lambda s_\alpha$, the overall scale is undetermined. Granting that $s_1(Z), s_2(Z), \ldots, s_N(Z)$ do not vanish simultaneously, this gives us a map to $\mathbb{P}^{N-1}$.

The simplest example is to embed $\mathbb{P}^1$ using $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(k)$ into $\mathbb{P}^k$. In this case the map is

$$i_k(Z_0, Z_1) = (Z_0^k,\ Z_0^{k-1}Z_1,\ Z_0^{k-2}Z_1^2,\ \ldots,\ Z_0 Z_1^{k-1},\ Z_1^k).$$

In general we want this map to be an embedding, i.e. that distinct points map to distinct points with non-vanishing Jacobian. In general, we can appeal to the Kodaira

[1] Or unless we can come up with a construction in which some sort of physical objects at the points p_i enforce the equations.

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embedding theorem, which asserts that for positive $\mathcal{L}$ this will be true for all $\mathcal{L}^k$ for some $k \geq k_0$. For non-singular quintics, this is true for $\mathcal{O}_M(k)$ for all $k \geq 1$. As a point of language, the pair of a manifold X with a positive line bundle $\mathcal{L}$ is referred to as a *polarized manifold* $(X, \mathcal{L})$; the condition that this construction provides an embedding for some k is that $\mathcal{L}$ is *ample*.

Now, we consider our family eq. (2.6) of candidate Kähler potentials, and rewrite them as

$$K_h = \log \left(\sum_{\alpha,\bar{\beta}} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}} \right)$$

or simply

$$K_h \equiv \log ||s||_h^2 \tag{2.8}$$

for short, where s_α plays the role of a degree k monomial. We now have an N^2-parameter family of Kähler potentials, and will seek a good approximating metric in this family. Just as before, this amounts to using the pull-back of a Fubini-Study metric from $\mathbb{P}^{N-1}$ as our trial metric.

Mathematically, the simplest interpretation of eq. (2.8) is that it defines a hermitian metric on the line bundle $\mathcal{L} = \mathcal{O}_M(k)$. This is a sesquilinear map from $\bar{\mathcal{L}} \otimes \mathcal{L}$ to smooth functions $C^\infty(X)$, here defined by

$$(s, s') = e^{-K_h} \cdot \bar{s} \cdot s' = \frac{\bar{s} \cdot s'}{\sum_{\alpha,\bar{\beta}} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}}.$$

The point is that a change of frame, which acts on our explicit sections as $s_\alpha \to \lambda s_\alpha$, cancels out of this expression.[2]

This metric allow us to define an inner product between the global sections:

$$H_{\alpha\bar{\beta}} = \langle s_\beta | s_\alpha \rangle = i \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{||s||_h^2} \, \mathrm{dvol}_X. \tag{2.9}$$

This is the "physical" inner product in a sense we will explain further below. Note that it depends on h in a nonlinear way, since h appears in the denominator.

Here $d\mathrm{vol}_X$ is a volume form on X, which has to be chosen. If X is Calabi-Yau, it is simplest to use eq. (2.7) to define $d\mathrm{vol}_X$. If X is not Calabi-Yau, the standard choice of $d\mathrm{vol}_X$ is to take $d\mathrm{vol}_\omega = \omega^n/n!$, where ω is the Kahler metric derived from eq. (2.8). This depends on h as well, so the expression is even more non-linear in h.

Thus, given h and a basis of global sections s_α, we could compute the matrix of inner products eq. (2.9). Once we have it, we could make a linear redefinition, say $\tilde{s} = H^{-1/2} s$, and go to a basis of orthonormal sections where

$$H_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}. \tag{2.10}$$

[2] A possibly more familiar physics use of this is in $N = 1$ supergravity: taking $K \to -K$ and $s \to W$, one gets the standard expression for the gravitino mass $e^K |W|^2$. In an example such as the flux superpotential, in which W is a sum of various terms s_α with constant coefficients, eq. (2.8) also applies to give K.

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On the other hand, eq. (2.8) also implicitly defines a notion of orthonormal basis locally in the bundle, in which

$$h^{\alpha\bar{\beta}} = \delta^{\alpha\bar{\beta}}. \tag{2.11}$$

This is *a priori* different from eq. (2.10); indeed we can freely postulate it when we write eq. (2.8). However, if the two notions agree,

$$H_{\alpha\bar{\beta}} = (h^{-1})_{\alpha\bar{\beta}},$$

then we can go to a basis of sections in which

$$H_{\alpha\bar{\beta}} = h^{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}. \tag{2.12}$$

In this case, the embedding of X in $\mathbb{P}^{N-1}$ using these sections is called *balanced*. More generally, we call a polarized manifold $(X, \mathcal{L}^k)$ balanced if such an embedding exists.

An equivalent definition of the balanced embedding is arrived at if we consider the function on X defined as

$$\rho(\omega)(x) = \sum_{\alpha,\bar{\beta}} (H^{-1})^{\alpha\bar{\beta}} (s_\alpha(x), \bar{s}_{\bar{\beta}}(x)) \tag{2.13}$$

or equivalently

$$\rho(\omega)(x) = \sum_{\alpha} ||s_\alpha(x)||^2$$

where the second sum is taken over an orthonormal basis in which $H = \delta_{\alpha\beta}$. X is balanced precisely when $\rho(\omega)(x)$ is the constant function.

Many theorems have been proven about balanced manifolds. Let us first recall the following theorem of Donaldson (Theorem 1 in [11]):

Theorem 1. *Suppose the automorphism group* $\mathrm{Aut}(X, \mathcal{L})$ *is discrete. If* $(X, \mathcal{L}^k)$ *is balanced, then the choice of basis in* $H^0(X, \mathcal{L}^k)$ *such that* $i_k(\mathcal{L})$ *is balanced is unique up to the action of* $\mathrm{U}(N) \times \mathbb{R}^*$.

The condition on $\mathrm{Aut}(X, \mathcal{L})$, i.e., there are no continuous symmetries, is true for the quintic Q. This theorem then tells us that, if a metric h exists which gives a balanced embedding, it is unique up to scale.

Given a balanced embedding, one defines the *balanced metric* on X as the pullback of the Fubini-Study metric (2.8):

$$\omega_k = \frac{2\pi}{k} i_k^*(\omega_{FS}), \tag{2.14}$$

The cohomology class of the Kahler form $[\omega_k] = 2\pi c_1(\mathcal{L}) \in H^2(X)$ is independent of k. Using these definitions Donaldson proves that (Theorem 2 in [11]):

Theorem 2. *Suppose* $\mathrm{Aut}(X, \mathcal{L})$ *is discrete and* $(X, \mathcal{L}^k)$ *is balanced for sufficiently large* k. *If the metrics* ω_k *converge in the* C^∞ *norm to some limit* ω_∞ *as* $k \to \infty$, *then* ω_∞ *is a Kahler metric in the class* $2\pi c_1(\mathcal{L})$ *with constant scalar curvature.*

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The constant value of the scalar curvature is determined by $c_1(X)$, and in particular for $c_1(X) = 0$ the scalar curvature is zero. Thus, the balanced metrics ω_k, in the large k limit, converge to the Ricci flat metric.

Therefore, if we can find the unique balanced metric for a given $\mathcal{L}$, it is a good candidate for approximating the Ricci flat metric on X. One may ask where the complex structure and Kahler moduli on which this Ricci flat metric depends, are put in. The complex structure enters implicitly through the basis for holomorphic sections s_α, as we will see in examples below. As for the Kahler class, recall that this is determined, up to scale, to be $2\pi c_1(\mathcal{L})$. Of course, the Ricci flatness condition is scale invariant, so the overall scale is irrelevant; however the point of this is that if $b^{1,1} > 1$, then by appropriately choosing $\mathcal{L}$ we choose a particular ray in the Kahler cone. This will not be relevant for our examples here but shows that in principle any Ricci-flat Kahler metric could be approximated in this way.

2.3 Finding the balanced metric

In [12, 4] Donaldson proposes a method to determine the hermitian metric h in eq. (2.8), which will lead to a balanced metric. He defines the "T operator", which given a metric h computes the matrix H:

$$H_{\alpha\bar{\beta}} = T(h)_{\alpha\bar{\beta}} \equiv \frac{N}{\mathrm{vol}(X)} \int_X \frac{s_\alpha \bar{s}_{\bar{\beta}}}{||s||_h^2} \, d\mathrm{vol}_X \tag{2.15}$$

Now, suppose we find a fixed point of this operator,

$$T(h) = h.$$

Then, by a $GL(N)$ change of basis $s \to h^{-1/2}s$, we can bring h to the unit matrix, which will produce the balanced embedding.

The simplest way to find a fixed point of an operator is to iterate it. If the operator is contracting, this is guaranteed to work. In our case we have the following theorem [11, 13]:

Theorem 3. *Suppose that* $\mathrm{Aut}(X, L)$ *is discrete. If a balanced embedding exists then, for any initial* G_0 *hermitian metric, the sequence* $T^r(G_0)$ *converges to the balanced metric* G *as* $r \to \infty$.

Thus the T operator can be used to find approximate Ricci-flat metrics on Calabi-Yau manifolds, and more generally approximate constant scalar curvature Kahler metrics. In [4] Donaldson studies numerically explicit $\mathbb{P}^1$ and $K3$ examples. We will discuss some additional examples below.

2.4 Balanced metrics and constant scalar curvature

In this subsection we outline the reason why the limit of a family of balanced metrics has constant scalar curvature. This is the content of Theorem 2. This will be very useful later on, when we generalize the T-operator to vector bundles.

Note that the function $\rho(\omega)$ is independent of the choice of orthonormal basis, and remains unchanged if we replace h by a constant scalar multiple. Therefore, it is an

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invariant of the Kahler form. As discussed before, the balanced condition for (X, L^k) is equivalent to the existence of a metric ω_k such that $\rho(\omega_k)$ is a constant function on X. The asymptotic behavior of the "density of states" function $\rho(\omega_k)$ as $k \to \infty$ for fixed ω has been studied in [10, 14 – 16]. Note that for any metric

$$\int_X \rho_k(\omega) = N = \dim H^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + \cdots , \tag{2.16}$$

where the coefficients a_i can be determined using the Riemann-Roch formula. Note that a_0 is just the volume of X and

$$a_1 = \frac{1}{2\pi} \int_X S(\omega),$$

where $S(\omega)$ is the scalar curvature of ω. We will use the following result (Prop. 6 in [11]):

Proposition 1. *1. $\rho(\omega)$ has an asymptotic expansion as $k \to \infty$*

$$\rho_k(\omega) \sim A_0(\omega) k^n + A_1(\omega) k^{n-1} + \cdots$$

where $A_i(\omega)$ are smooth functions on X defined locally by ω. In particular,

$$A_0(\omega) = 1, \qquad A_1(\omega) = \frac{1}{2\pi} S(\omega).$$

2. The expansion holds uniformly in the C^∞ norm; in that for any $r, N > 0$

$$\left\| \rho_k(\omega) - \sum_{i=0}^{N} A_i(\omega) k^{n-i} \right\|_{C^r(X)} \leqslant K_{r,N,\omega} k^{n-N-1}$$

for some constants $K_{r,N,\omega}$.

Now assume that we are given balanced metrics ω_k converging to ω_∞. Then by the previous proposition

$$\left\| \rho_k(\omega_k) - k^n - \frac{1}{2\pi} S(\omega_k) k^{n-1} \right\|_{C^0(X)} \leqslant c k^{n-2}$$

for some constant c. Since ω_k is balanced $\rho_k(\omega_k)$ is constant: $\rho_k(\omega_k) = \dim H^0(X, L^k)/V$, and we can use (2.16) to find that

$$\left\| \frac{1}{V}(V k^n + a_1 k^{n-1} + \cdots) - k^n - \frac{1}{2\pi} S(\omega_k) k^{n-1} \right\|_{C^0(X)} \leqslant c k^{n-2},$$

or equivalently

$$\left\| \frac{2\pi}{V} a_1 - S(\omega_k) \right\|_{C^0(X)} = O(k^{-1})$$

Hence $S(\omega_\infty) = S_0 = constant$, where $S_0 = \frac{1}{V} \int_X S(\omega)$ is the mean curvature.

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3. The hermitian Yang-Mills equations

We are now ready to generalize the T-operator, which provided an approximation scheme for the constant curvature metric, to a "generalized T-operator" which can be used to find a solution of the Yang-Mills equations on a Calabi-Yau manifold X.

We briefly recall the argument that a solution of the Yang-Mills equations which preserves $\mathcal{N} = 1$ supersymmetry, must be hermitian Yang-Mills. First, the supersymmetry variation of the gaugino has to vanish,

$$\Gamma^{\mu\nu} F^a_{\mu\nu} \epsilon = 0,$$

where $F^a_{\mu\nu}$ is the Yang-Mills field strength, and ϵ is the covariantly constant spinor.

Going to complex coordinates $(i, \bar{i})$ and rewriting of the Clifford algebra as

$$\Gamma_i \to dz^i; \qquad \Gamma_{\bar{i}} \to \omega_{\bar{i}j}\, \partial^j,$$

this is equivalent to

$$F_{ij} = F_{\bar{i}\bar{j}} = 0; \qquad \omega^{i\bar{j}} F_{i\bar{j}} = 0.$$

This is the particular case of the hermitian Yang-Mills equations with $\mathrm{Tr}\, F = 0$. The general case replaces the last equation with

$$\omega^{i\bar{j}} F_{i\bar{j}} = c \cdot \mathbf{1}$$

for a constant c, determined by the first Chern class. For convenience we abbreviate this equation below as

$$\bigwedge F = c \cdot \mathbf{1}.$$

Next we review the relation between solutions of these equations, and holomorphic bundles carrying hermitian-Einstein metrics. In physics, one defines Yang-Mills theory in terms of a connection on a vector bundle with a fixed metric. First, a connection on a vector bundle can be described in terms of a connection one-form by choosing a frame for the bundle, say $e_a(x)$, and defining the covariant derivative as

$$D(v^a e_a) = (dv^a) e_a + v^a A^b_a e_b.$$

In physics, one usually takes the frame to be orthonormal, $(e_a, e_b) = \delta_{ab}$, and thus

$$(u, v) = (u^a e_a, v^b e_b) = (u^a)^* v^a, \tag{3.1}$$

where $*$ is complex conjugation.

The condition that the connection be compatible with the metric,

$$d(u, v) = (Du, v) + (u, Dv), \tag{3.2}$$

reduces to requiring the connection one-form to be anti-hermitian,

$$A^{(phys)}{}_i = -A^{(phys)\dagger}_i. \tag{3.3}$$

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In mathematics, one often considers a more general frame, for which the metric is a hermitian matrix,

$$(e_a, e_b) = G_{\bar{a}b}, \qquad G = G^\dagger. \tag{3.4}$$

Decomposing the positive definite hermitian matrix G as

$$G = h^\dagger h, \tag{3.5}$$

we see that the math and physics conventions differ by a complex gauge transformation: $u = h\, s$. This complex gauge transformation leads to a different form for the connection, according to the standard relation

$$\partial_i + A^{(math)}{}_i = h(\partial_i + A^{(phys)}{}_i)h^{-1}. \tag{3.6}$$

Now, equations of the form

$$F_{\bar{i}\bar{j}} = 0 \ \forall\ \bar{i}, \bar{j}$$

will be integrability conditions for the covariant derivatives. In particular, this equation has the general solution

$$\partial_{\bar{i}} + A^{(phys)}{}_{\bar{i}} = g^{-1}\bar{\partial}_{\bar{i}} g,$$

in other words the $\bar{D}$ covariant derivatives are obtained from the derivative $\bar{\partial}$ by a complex gauge transformation.

Thus, we can use eq. (3.6) to bring the connection to the gauge $\bar{A}^{(math)} = 0$, at the cost of losing the simple metric eq. (3.1) and eq. (3.3). Actually, the covariant derivative is still compatible with the metric as in eq. (3.2), we just have a non-trivial fiber metric G. The metric compatibility condition becomes

$$0 = \partial(u, v) = (\bar{\partial} u, v) + (u, Dv)$$

so

$$\partial G_{\bar{a}b} = G_{\bar{a}c} A^{(math)}{}^c_b$$

or equivalently

$$A^{(math)} = G^{-1}\partial G.$$

Conversely, if we are given a metric G, then we can use the inverse complex gauge transformation to bring the connection back to the unitary form. This leads to the formula

$$\bar{A}^{(phys)}_{\bar{i}} = h(\bar{\partial}_{\bar{i}} h^{-1}).$$

Using eq. (3.3), we can get the entire connection, so the metric contains the same information as a connection satisfying $F^{(0,2)} = F^{(2,0)} = 0$. Thus we can rephrase the final equation on $F^{(1,1)}$, as a condition on the metric. It is simplest to write this in the "mathematical" gauge $\bar{A}^{(math)} = 0$, in which it is

$$c \cdot \mathbf{1} = \omega^{i\bar{j}} F_{i\bar{j}} = \omega^{\bar{j}i}\bar{\partial}_{\bar{j}} A^{(math)}{}_i = \omega^{\bar{j}i}\bar{\partial}_{\bar{j}}\left(G^{-1}\partial_i G\right). \tag{3.7}$$

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A metric G satisfying this equation is a "hermitian-Einstein" metric. It is simply related to a hermitian Yang-Mills connection as above.

Finally, using the complex gauge transformation above, the standard physical inner product

$$\langle u|v\rangle \equiv \int_X (u^*)_a v^a \tag{3.8}$$

is equal to the natural inner product generalizing eq. (2.9),

$$\langle u|v\rangle = \langle h\ s|h\ t\rangle \tag{3.9}$$

$$= i\int_X G_{a\bar{b}}\ \bar{s}^{\bar{b}}\ t^a\, d\mathrm{vol}_X. \tag{3.10}$$

3.1 Embedding vector bundles

We now want to represent the hermitian metric $G_{a\bar{b}}$ in the same way as we did for line bundles, by introducing a complete basis of sections. Now an irreducible bundle E with $c_1 = 0$, and thus of interest for string compactification, will not have global sections. What we do instead is to make the same construction for $E(k) \equiv E \otimes \mathcal{L}^k$, which will have global sections. We can again think of these sections as a basis of polynomials approximating functions on which to base our numerical scheme.

Thus, consider a rank r vector bundle E, and suppose that $E(k)$ has N global sections. Choosing a local frame as above, a basis for these will be an N by r matrix z^a_α. This is defined up to a $GL(N)$ change of basis, and up to a $GL(r)$ change of frame. After making these identifications, such a matrix z defines a point in the Grassmannian $G(r, N)$ of r planes in $\mathbb{C}^N$.

Given a metric $G_{a\bar{b}}$ on the fibers of $E(k)$, we can define the matrix of inner products

$$H_{\alpha\bar{\beta}} = \langle z_\beta | z_\alpha\rangle$$

as above. Such a metric could be obtained by multiplying a metric $G^{(0)}$ on E, by one on $\mathcal{L}^k$ as defined earlier. Or, it might simply be an $r \times r$ hermitian matrix of functions (in each local frame) with appropriate transformation properties.

Now there is a natural set of metrics on $E(k)$ generalizing eq. (2.8), again parameterized by an $N \times N$ matrix, defined by

$$(G^{-1})^{a\bar{b}} = g^{\alpha\bar{\beta}} z^a_\alpha (z^\dagger)^{\bar{b}}_{\bar{\beta}}\,,$$

where the dagger is hermitian conjugation. Again, the approach will be to find a natural metric in this class which is a good approximation to the hermitian-Einstein metric. This will lead to a hermitian Yang-Mills connection on $E(k)$. But this is simply related to the hermitian Yang-Mills connection on E, because twisting by $\mathcal{L}^k$ only modifies the trace part of the field strength.

3.2 Generalized T-operator

We will now turn to a proposal for a generalized T-operator, which produces the hermitian-Einstein metric on a stable vector bundle. To begin with we use results by Wang about balanced metrics on such bundles [5].

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We consider again a polarized n dimensional manifold $(X, \mathcal{L})$ and an irreducible holomorphic vector bundle E of rank r on X. Then by Kodaira embedding we know that for k sufficiently large, a basis z^a_α of the global sections of $E(k)$ will give rise to an embedding

$$X \xhookrightarrow{i} G(r, N).$$

Now Wang proves the following:

Theorem 4. *E is Gieseker stable iff there is an integer k_0 such that for $k > k_0$, the* kth *embedding given as above can be moved to a balanced place, i.e., there is a $g \in \mathrm{SL}(N, \mathbb{C})$ which is unique up to left translation by* $\mathrm{SU}(N)$ *such that:*

$$\frac{1}{V}\int_{g\cdot X} z(z^\dagger z)^{-1}z^\dagger \; dV = \frac{r}{N} I_{N\times N}.$$

We call the equation above the "balance equation." In the case that E is a line bundle, this definition reduces to that of a balanced embedding in $\mathbb{P}^{N-1}$.

Now, let h be a hermitian metric on $\mathcal{L}$ and H be a hermitian metric on E, and fix the Kähler form on X to be $\omega = \frac{i}{2\pi} Ric(h)$. Let vol denote the volume of (X, ω). Suppose $S_1, \ldots, S_N$ is an orthonormal basis of $H^0(X, E(k))$ with respect to the induced L^2 -metric $\langle .,. \rangle$. The Szegö kernel B_k is a generalization of the function $\rho(\omega)$ defined in eq. (2.13). It is defined as the fiberwise homomorphism

$$B_k(x) = \sum_{i=1}^{N} \langle ., S_i(x)\rangle S_i(x) \colon E_x \to E_x.$$

This expression is independent of the choice of orthonormal basis.

Now the local form of Theorem 4 can be stated as follows (Corollary 1.1 of [5]):

Theorem 5. *E is Gieseker stable iff there is an integer k_0 such that for any $k > k_0$, we can find a metric $H^{(k)}$, which we will call the* balanced metric *on $E(k)$, such that the Szegö kernel satisfies the equation*

$$B_k(x) = \frac{\chi(k)}{V\ r} I_E$$

where I_E is the identity bundle morphism and $\chi(k)$ is the Hilbert polynomial of E with respect to the polarization $\mathcal{L}$.

The theorem tells us that if E is Gieseker stable then for large k there is a balanced metric $H^{(k)}$ on $E(k)$. Hence we will have a sequence of hermitian metrics $H_k := H^{(k)} \otimes h^{-k}$ on E. The importance of the balanced metric $H^{(k)}$ for physical applications follows from the following theorem:

Theorem 6. *Suppose E is Gieseker stable. If $H_k \to H_\infty$ in the C^∞ norm as $k \to \infty$, then the metric H_∞ solves the "weak hermitian-Einstein equation",*

$$\frac{i}{2\pi}\bigwedge F_{(E,H_\infty)} + \frac{1}{2}S(\omega)I_E = \left(\frac{deg(E)}{Vr} + \frac{\bar{s}}{2}\right)\cdot I_E \tag{3.11}$$

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where $\bigwedge F_{(E,H_\infty)}$ *is the contraction of the curvature form of* E *with* ω, $S(\omega)$ *is the scalar curvature of* X *and* $\bar{s} := \frac{1}{V}\int_X S\frac{\omega^n}{n!}$. *Conversely, suppose there is a hermitian metric* H_∞ *solving this equation, then* $H_k \to H_\infty$ *in* C^r *norm for any* r.

To prove (3.11) one can work along the same lines as in the proof of Theorem 2, using Catlin's and Wang's results for the expansion of the Szegö kernel.

Proposition 2. *1. For fixed hermitian metrics* H *and* h *on* E *and* $\mathcal{O}_X(1)$ *respectively, there is an asymptotic expansion as* $k \to \infty$

$$B_k(H,h) \sim A_0(H,h)k^n + A_1(H,h)k^{n-1} + \cdots ,$$

where $A_i(H,h) \in \Gamma(\text{ End } E)$ *are smooth sections defined locally by* H. *In particular,*

$$A_0(H,h) = I_E, \quad A_1(H,h) = \frac{i}{2\pi}\bigwedge F(E, Ric(h)) + \frac{1}{2}S(\omega)\cdot I_E$$

2. The expansion holds uniformly in the C^∞ *norm; in the sense that for any* $r, N > 0$

$$\left\| B_k(H,h) - \sum_{i=0}^{N} A_i(H,h)k^{n-i} \right\|_{C^r} \leqslant K_{r,N,H,h}k^{n-N-1}$$

for some constants $K_{r,N,H,h}$.

Now we can repeat the steps of the argument outlined in section 2.4. Under the assumption that $H_k \to H_\infty$ in C^∞ we find that for $r > 0$

$$\left\| B_k(H_k) - I_E k^n - \frac{i}{2\pi}\bigwedge F(E, Ric(h)) + \frac{1}{2}S(\omega)\cdot I_E k^{n-1} \right\|_{C^r} \leqslant Ck^{n-2}$$

for some fixed constant C. By assumption $H^{(k)}$ is balanced, hence $B_k(H_k) = \chi(k)/rVI_E$. This implies that

$$\left\| \frac{i}{2\pi}\bigwedge F_{(E,H_\infty)} + \frac{1}{2}S(\omega)I_E - \left(\frac{deg(E)}{Vr} + \frac{\bar{s}}{2}\right)\cdot I_E \right\| = O(k^1).$$

3.2.1 Generalized T-operator

Using the strong analogy between the construction of metrics with constant Kahler curvature and metrics on stable bundles which obey the hermitian-Einstein equation, we propose the following generalized T-operator:

$$T(G) = \frac{N}{Vr}\int_X z(z^\dagger G^{-1}z)^{-1}z^\dagger \, dV, \tag{3.12}$$

where as before, z is an N by r matrix of holomorphic sections of E.

The relevance of this proposal follows from the following conjecture:

Conjecture 1. *If a balanced embedding* $i\colon X \hookrightarrow G(r,N)$ *exists, then for every hermitian* $N \times N$ *matrix* G, *the sequence* $T^r(G)$ *converges to a fixed point* G_0 *as* $r \to \infty$. *Using an orthonormal basis with respect to* G_0, *the embedding is balanced, and as outlined above, it provides an approximate solution for the corresponding hermitian-Einstein equation.*

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This conjecture may require additional technical assumptions, such as the earlier one of $\mathrm{Aut}(X, E)$ being discrete. We have not attempted to prove it, but would hope that this can be done along the same lines as [11, 13].

In the following section we will numerically test the conjecture for several stable vector bundles on $\mathbb{P}^2$, and on the Fermat quintic in $\mathbb{P}^4$, and find that it works for these cases.

4. Examples

4.1 Hermite-Einstein metric on the tangent bundle of $\mathbb{P}^n$

Let $\mathbb{P}^n$ be the complex projective space of dimension n, and $\{Z_i\}_{i=0}^{i=n}$ its homogeneous coordinates. We will work on the open set $Z_0 \neq 0$ and chose the local inhomogeneous coordinates $w_i = Z_i/Z_0$. The Fubini-Study metric on $\mathbb{P}^n$

$$g_{i\bar{j}} = \frac{1}{1+\sum_i |w_i|^2}\delta_{i\bar{j}} - \frac{w_i \bar{w}_j}{(1+\sum_i |w_i|^2)^2}.$$

is the unique maximally symmetric metric, with its group of Killing symmetries isomorphic to $\mathrm{U}(n+1)$. In addition, this metric is Einstein, that is its Ricci tensor is proportional to the metric itself. Therefore its associated curvature tensor obeys the hermitian Yang-Mills equation. The Donaldson-Uhlenbeck-Yau theorem then implies that the tangent bundle of $\mathbb{P}^n$, $T\mathbb{P}^n$, is a rank n stable bundle on $\mathbb{P}^n$.[3] It follow from this that the balanced metric on the bundle $T\mathbb{P}^n$ must be the Fubini-Study metric.

To describe the tangent bundle $T\mathbb{P}^n$ we use the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{P}^n \longrightarrow 0. \tag{4.1}$$

Here $\mathcal{O}_{\mathbb{P}^n}(1)$ denotes the hyperplane line bundle. After twisting the sequence by $\mathcal{O}_{\mathbb{P}^n}(k)$ and taking the cohomology we find the short exact sequence (SES)

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k+1)^{\oplus(n+1)}) \longrightarrow H^0(\mathbb{P}^n, T\mathbb{P}^n(k)) \longrightarrow 0. \tag{4.2}$$

This gives an explicit description for $H^0(\mathbb{P}^n, T\mathbb{P}^n(k))$, which for sufficiently large k gives the embedding

$$\mathbb{P}^n \hookrightarrow \mathrm{G}(n,\, W) \tag{4.3}$$

where $W = H^0(\mathbb{P}^n, T\mathbb{P}^n(k))^*$, and $\mathrm{G}(n,\, W)$ is the Grassmannian of n-planes in W.

Based on (4.2), we choose to describe the global sections of $T\mathbb{P}^n(k)$ by an $n+1$ vector

$$(M_0, \ldots, M_n)$$

where $\{M_i\}_{i=1}^n$ are arbitrary monomials of degree $k+1$ in the homogeneous coordinates Z_i, while M_0 is any degree $k+1$ monomial which does not contain an Z_0.

Now we show how to construct the embedding (4.2) for any $k \geq 0$. We start by choosing a frame $\{\hat{e}_i\}_{i=0}^n$ for the vector bundle $\mathcal{O}(k+1)^{\oplus(n+1)}$. This amounts to choosing

[3] The stability of $T\mathbb{P}^n$ also has purely algebraic proof.

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a section for every one the $n+1$ summands. For simplicity we chose the same section in every summand. The Euler sequence (4.1) imposes the condition

$$\sum_{i=0}^{n} Z_i \hat{e}_i = 0.$$

Locally this gives a frame for $T\mathbb{P}^n$, if we solve for

$$\hat{e}_0 = -\sum_{i=1}^{n} \frac{Z_i}{Z_0} \hat{e}_i = -\sum_{i=1}^{n} w_i \hat{e}_i.$$

Expanding the global sections of $T\mathbb{P}^n(k)$ in the local frame $\{\hat{e}_i\}_{i=1}^n$ gives an $n \times \dim(W)$ matrix, which is the explicit realization of our embedding [17].

To illustrate the procedure consider $T\mathbb{P}^2(0)$. $\mathcal{O}_{\mathbb{P}^2}(1)$ has 3 global sections: Z_0, Z_1, Z_2. Choosing Z_0 to be the local frame in every summand of $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$, and discarding the global section Z_0 from the first $\mathcal{O}_{\mathbb{P}^2}(1)$, we find the matrix

$$z = \begin{pmatrix} -w_1^2 & -w_1 w_2 & 1 & w_1 & w_2 & 0 & 0 & 0 \\ -w_1 w_2 & -w_2^2 & 0 & 0 & 0 & 1 & w_1 & w_2 \end{pmatrix} \tag{4.4}$$

For an initial hermitian metric G_0 on the vector space $W = H^0(\mathbb{P}^n, T\mathbb{P}^n(k))^*$, our generalized T-operator (3.12) gives the iterations

$$G_{m+1} = T(G_m) = \frac{\dim W}{n \operatorname{Vol}(\mathbb{P}^n)} \int_{\mathbb{P}^n} z(z^\dagger G_m^{-1} z)^{-1} z^\dagger \, dV.$$

We tested the converges of the T-map starting with $G_0 = I$ in the case $n = 2$ for $k = 1, \ldots, 5$. In all cases we converged to a given G_∞ for less than 10 iterations, with a precision of 0.1%.

The balanced metric $H^{(k)}$ on the vector bundle $T\mathbb{P}^n(k)$ induced by G_∞ is given by

$$H^{(k)} = (z^\dagger G_\infty^{-1} z)^{-1}. \tag{4.5}$$

Let h be Fubini-Study metric on the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$, that is the metric with constant scalar curvature. Then the metric

$$H_k := H^{(k)} \otimes h^{-k} = (z^\dagger G_\infty^{-1} z)^{-1} \otimes h^{-k}$$

is the balanced metric on $T\mathbb{P}^n$. Our numerical computations show that this is indeed the Fubini-Study metric on $T\mathbb{P}^n$, as explained earlier. The numerical agreement is within 0.5%. This provides the first non-trivial test of our conjecture.

4.2 A stable rank 3 bundle over $\mathbb{P}^2$

In this section we test our generalized T-operator on a rank 3 vector bundle V^* over $\mathbb{P}^2$. We first consider its dual V, defined by four linearly independent global sections $\{m_i\}$ of $\mathcal{O}_{\mathbb{P}^2}(2)$ through the SES

$$0 \longrightarrow V \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \xrightarrow{m} \mathcal{O}_{\mathbb{P}^2}(2) \longrightarrow 0.$$

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This bundle has moduli, which are implicitly determined by the choice of the sections $\{m_i\}$. Before choosing these, let us check stability, which does not depend on the specifics of this choice.

To check stability, we have to ensure that neither V nor $\wedge^2 V$ have destabilizing line bundles. Using the canonical isomorphism

$$\wedge^2 V = \det V \otimes V^*$$

we find the slopes

$$\mu(V) = -2/3, \;\; \mu(\wedge^2 V) = -4/3.$$

Since $Pic(\mathbb{P}^2) = \mathbb{Z}$, all line bundles are of the form $\mathcal{O}_{\mathbb{P}^2}(p)$ for some p. Hence it is sufficient to show that

$$H^0(\mathbb{P}^2, V) = 0, \quad H^0(\mathbb{P}^2, \wedge^2 V(1)) = 0.$$

The first fact follows from the defining sequence of V, if we assume that $\{m_i\}$ are linearly independent. To prove the second statement we use

$$H^0(\mathbb{P}^2, \wedge^2 V(1)) = H^0(\mathbb{P}^2, V^*(-1)) = H^2(\mathbb{P}^2, V(-2))^*.$$

Again, this statement follows easily from the defining sequence of V. Finally, stability of V implies stability for V^*.

We will now compute the hermitian Yang-Mills connection on V^* using our generalized T-operator. First observe that $V^*(k)$ fits into the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(k-2) \to \mathcal{O}_{\mathbb{P}^2}(k)^{\oplus 4} \to V^*(k) \to 0. \tag{4.6}$$

This leads to another SES

$$0 \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-2)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)^{\oplus 4}) \to H^0(\mathbb{P}^2, V^*(k)) \to 0.$$

We can use this expression for an explicit parameterization of $H^0(\mathbb{P}^2, V^*(k))$.

For concreteness let us choose to be four global sections $\{m_i\}_{i=1}^4$ defining V to be

$$Z_1 Z_2, \; Z_0 Z_1, \; Z_0 Z_2, \; Z_0^2.$$

Now we choose a frame $\{\hat{e}_i\}$ for $\mathcal{O}_{\mathbb{P}^2}(k)^{\oplus 4}$. The defining equation (4.6) of $V^*(k)$ imposes the condition $\sum_i m_i \hat{e}_i = 0$, and gives a frame for V^*. Locally we can solve for $\hat{e}_0$, and working in inhomogeneous coordinates w_i we find that

$$\hat{e}_0 = -\frac{1}{w_2}\hat{e}_1 - \frac{1}{w_1}\hat{e}_2 - \frac{1}{w_1 w_2}\hat{e}_3.$$

Expanding the global sections of $H^0(\mathbb{P}^2, V^*(k))$ in the frame $\{\hat{e}_i\}_{i=1}^3$ gives a matrix, which is the embedding map.

We studied the convergence of our generalized T-operator numerically for $k = 2, 3$ and 4, and found that convergence was achieved for less than 10 iterations. As before, the metric on $V^*(k)$ is given by

$$H^{(k)} = (z^\dagger G_\infty^{-1} z)^{-1}, \tag{4.7}$$

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while the corresponding metric on V^* is

$$H_k := H^{(k)} \otimes h^{-k} = (z^\dagger G_\infty^{-1} z)^{-1} \otimes h^{-k},$$

where h is again the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^2}(1)$.

Since in this case the balanced metric on V^* is not a priori known, one needs a different approach, than used in the previous section for $T\mathbb{P}^2$, to test how close is the approximate balanced metric to satisfying the hermitian Yang-Mills equation. But this quite easy to do numerically once the balanced metric G_∞ is known, as all we need to do is to check how close we are to satisfying eq. (3.7). In all cases considered eq. (3.7) was satisfied to within 1% accuracy.

4.3 A rank 3 bundle on the Fermat quintic

In this section we turn to a much more complicated case than our previous examples, that of a stable rank 3 bundle on the Fermat quintic Q in $\mathbb{P}^4$:

$$Q: \qquad Z_0^5 + Z_1^5 + \cdots + Z_4^5 = 0. \tag{4.8}$$

Testing our generalized T-operator in this case necessitates knowledge of the Ricci flat metric on the Fermat quintic. For this we use Donaldson's original T-operator eq. (2.15), which we turn to first.

4.3.1 Ricci flat metric on Fermat quintic

We consider the embedding of Q given by the complete linear system of cubics, $H^0(Q, \mathcal{O}_Q(3))$, whose complex projectivization is isomorphic to $\mathbb{P}^{34}$. The balanced metric will be the restriction of a Fubini-Study metric on $\mathbb{P}^{34}$. An indirect test that this has indeed vanishing Ricci curvature is included in the next section.

In order to do practical calculations with Donaldson's T-map, we have to perform the integrals on Q numerically. We introduce a discrete approximation to the Calabi-Yau volume form $d\mu_\Omega = \Omega \wedge \bar{\Omega}$, defining it by a weighted set of M points $\{x_a\}_{a=1}^M \in Q$, with masses ν_a:

$$\int_Q (\ )\, d\mu_\Omega \approx \sum_{a=1}^{M} (\ )\, \delta(x - x_a)\nu_a. \tag{4.9}$$

This numerical measure gives an accurate approximation to the analytical one for large M. In our computations we build 10 different samples of 100,000 points, which we use independently to iterate the T-map until convergence is reached, i.e., the sequence $\{T^r(G_0)\}_{r=0}$ obeys

$$||T^{r+1}(G_0) - T^r(G_0)|| < \epsilon.$$

In our simulations the fixed point of this discrete version of the T-map was reached after 15-20 iterations. Each weighted point set gave rise to a convergent sequence. The 10 different hermitian forms $\{G_\infty^e\}_{e=1}^{10}$ approximating the balanced metric in $\mathbb{P}H^0(Q, \mathcal{O}_Q(3))$ agree up to

$$\max \left[\frac{\sigma(G_\infty^e)}{|\langle G_\infty^e \rangle|} \right] \approx 0.9\%, \tag{4.10}$$

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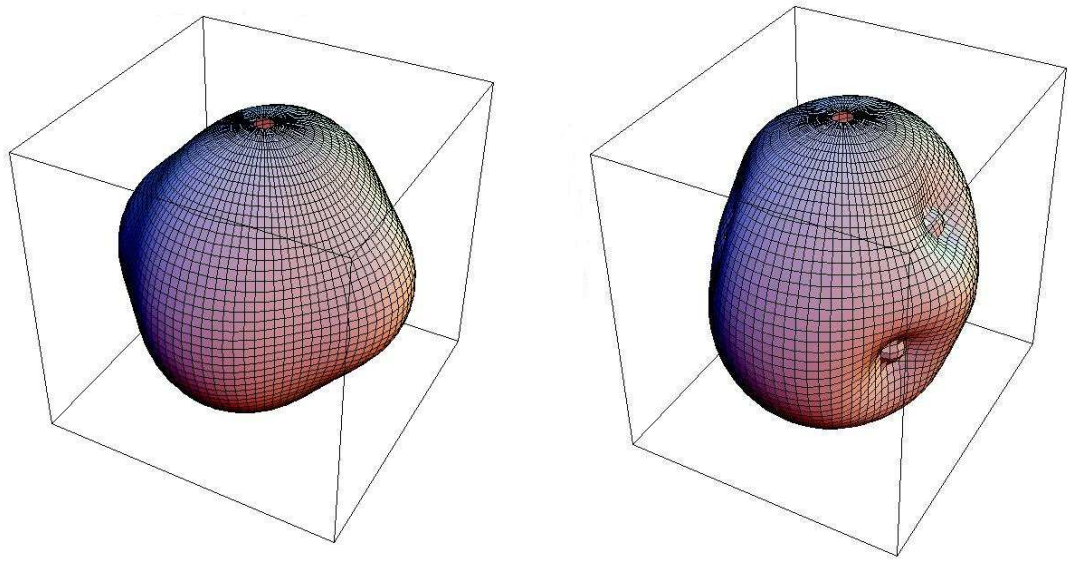

Figure 1: The shape of the rational curve for the balanced and non-balanced metrics.

where $|\langle G^e_\infty\rangle|$ is the average matrix of the ten different outputs and $\sigma(G^e_\infty)$ is the standard deviation matrix. The ratio $\sigma(G^e_\infty)/|\langle G^e_\infty\rangle|$ is computed entry by entry, and the maximum is taken over all entries. We used the average $\langle G^e_\infty\rangle$ as approximation for the balanced metric on $H^0(Q, \mathcal{O}_Q(3))$.

To get a visual picture of the geometry implicit in the construction, we consider the rational curve $t\colon \mathbb{P}^1 \hookrightarrow Q$, defined locally by the parametrization

$$\Big(1,\ -1,\ t,\ 0,\ -t\Big) \tag{4.11}$$

with $t \in \mathbb{C} \cup \infty$. Take the sections $Z_1^3 + Z_4^3$ and Z_0^3 from $H^0(Q, \mathcal{O}_Q(3))$, and consider the function $s = (Z_1^3 + Z_4^3)/Z_0^3 = w_1^3 + w_4^3$. In figure 1 we consider the real function $|s|^2_G$ restricted to the rational curve (4.11), where we take the stereographic projection of the complex t-plane and for a given t we plot $|s(t)|^2_G$ in the radial direction. For the balanced metric $\langle G^e_\infty\rangle$ the deviation from being spherical is small. For comparison we also show the same plot for the case of a generic non-balanced hermitian form G with random entries.

4.3.2 Solution of the hermitian Yang-Mills equation

In this section we use the generalized T-operator to produce a hermitian Yang-Mills connection on a rank three stable vector bundle V on the Fermat quintic Q. We also implicitly test that the previously obtained balanced metric on Q indeed has vanishing Ricci curvature.

We define the rank three bundle V by the following SES

$$0 \longrightarrow \mathcal{O}_Q(-1) \xrightarrow{\ \beta\ } \mathcal{O}_Q^{\oplus 4} \longrightarrow V \longrightarrow 0.$$

β is given by four generic global sections of $\mathcal{O}_Q(1)$, which do not intersect on Q, hence V is indeed a vector bundle. In addition, the first Chern class of V is $c_1(V) = H$, hence V

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is not a simple twist of the tangent bundle of Q. That fact that V is stable was proved in [18].

Once again, we use

$$0 \longrightarrow \mathcal{O}_Q(k-1) \longrightarrow \mathcal{O}_Q^{\oplus 4}(k) \longrightarrow V(k) \longrightarrow 0,$$

and its associated long exact sequence in cohomology

$$0 \longrightarrow H^0(Q, \mathcal{O}_Q(k-1)) \longrightarrow H^0(Q, \mathcal{O}_Q^{\oplus 4}(k)) \longrightarrow H^0(Q, V(k)) \longrightarrow 0,$$

to derive a frame for V and an explicit parameterization for the global sections. We choose $\beta = (Z_0, \ldots, Z_3)$. Using the frame $\{\hat{e}_i\}_{i=0}^4$ for $\mathcal{O}_Q^{\oplus 4}$, we also get a frame for V with the relation

$$\hat{e}_0 = -\sum_{i=1}^{3} w_i \hat{e}_i.$$

In this paper we restrict to the case $k = 1$ for which $\dim H^0(Q, V(1)) = 19$. The coordinate matrix

$$z(w) = \begin{pmatrix} 1\ldots w_4 & 0 & 0 & -w_1^2 & -w_1w_2 & -w_1w_3 & -w_1w_4 \\ 0 & 1\ldots w_4 & 0 & -w_1w_2 & -w_2^2 & -w_3w_2 & -w_4w_2 \\ 0 & 0 & 1\ldots w_4 & -w_1w_3 & -w_2w_3 & -w_3^2 & -w_4w_3 \end{pmatrix} \tag{4.12}$$

gives the embedding into the Grassmannian $Q \hookrightarrow G(3, 19)$.

Using the integration techniques described in the previous section, we iterate the generalized T-operator. We reach the fixed point after 12-15 iterations for several different samples of weighted points which approximate the analytical measure, allowing us to estimate the balanced metric for $H^0(Q, V(1))$ with an error of 1.1%.[4]

The metric on $V(1)$ is given by

$$H = (z^\dagger G_\infty^{-1} z)^{-1}, \tag{4.13}$$

To test the accuracy of this metric we evaluate the right hand side of the hermitian Yang-Mills equations (3.7). We find the mean to be

$$\langle \omega^{i\bar{j}} F_{i\bar{j}} \rangle = \frac{1}{\mathrm{Vol}(Q)} \int_Q \left(\omega^{i\bar{j}} F_{i\bar{j}}\right) d\mu_\Omega \approx 1.31 \times \mathbf{I}_{3\times 3}$$

with $\mathbf{I}_{3\times 3}$ the 3×3 identity matrix. In our conventions the theoretical value of the constant is 4/3. The standard deviation of the individual matrix elements is

$$\sigma\left(\omega^{i\bar{j}} F_{i\bar{j}}\right) = \max\left[\sqrt{\frac{1}{\mathrm{Vol}(Q)} \int_Q \left(\omega^{i\bar{j}} F_{i\bar{j}} - \langle \omega^{i\bar{j}} F_{i\bar{j}} \rangle\right)^2 d\mu_\Omega}\,\right] \approx 0.15,$$

where the square-root and the square are performed entry by entry. Therefore, $\omega^{i\bar{j}} F_{i\bar{j}}$ is a global constant on Q times the identity, within an error of $0.15/1.31 \approx 11\%$. This implies that the hermitian Yang-Mills equation (3.7) is satisfied with this accuracy.

[4] We estimate the errors using (4.10).

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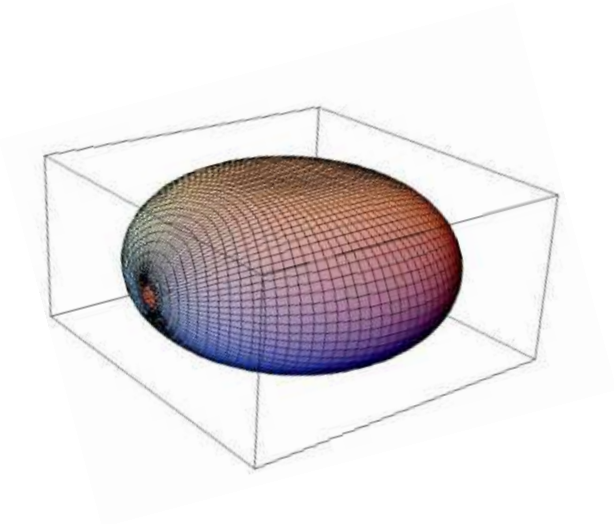

Figure 2: The probability density in the radial direction on the rational curve.

Testing the hermitian Yang-Mills equation provides an implicit test of Ricci flatness, since it is precisely the Ricci flat metric that is needed in the hermitian Yang-Mills equation. If we had gotten this metric wrong, then we would have had no chance of satisfying the hermitian Yang-Mills equation.

Finally, to visualize the construction, in figure 2 we took the rational curve defined in (4.11), and we plotted the function $|\Psi|^2_G$, where

$$\Psi = (w_1 w_4)\hat{e}_1 + (w_2 w_4)\hat{e}_2 + (w_3 w_4)\hat{e}_3$$

and G is the balanced metric we obtained. If we interpret Ψ as a wave-function, then figure 2 exhibits the probability density $\langle\Psi|\Psi\rangle$ in the radial direction, restricted to the rational curve (4.11).

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